Math 564: Advance Analysis 1
Lecture 2

Examples (continued). ○ In any metric (more generally, topological) space, clopen sets form an algebra.

○ In any set X, the finite and co-finite (= complement is finite) subsets form an algebra, while the countable and co-countable subsets form a σ-algebra.

Generation of algebras and σ-algebras.

Observation. Arbitrary intersections of algebras (resp. σ-algebras) is an algebra (resp. σ-algebra). More precisely, for a set X, if $\mathcal{A}_i \subseteq \mathcal{P}(X)$ is an algebra (resp. σ-algebra) for each $i \in I$, then $\bigcap_{i \in I} \mathcal{A}_i$ is also an algebra (resp. σ-algebra).

(Here, I is an index set, which may be unctbl.)

Thus, for a given collection $\mathcal{C} \subseteq \mathcal{P}(X)$, we may define the

○ algebra generated by $\mathcal{C}$: $\langle \mathcal{C} \rangle := \bigcap_{\substack{\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{P}(X) \\ \mathcal{A} \text{ algebra}}} \mathcal{A}$.

○ σ-algebra generated by $\mathcal{C}$: $\langle \mathcal{C} \rangle_\sigma := \bigcap_{\substack{\mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{P}(X) \\ \mathcal{S}\ \sigma\text{-algebra}}} \mathcal{S}$.

These are top-down definitions, using which would be hard to compute even $\langle \mathcal{C} \rangle$. We now give bottom-up / constructive equivalents.

Proposition. Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$.

(a)
$$\langle \mathcal{C} \rangle = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n, \quad \text{where } \mathcal{C}_0 := \mathcal{C} \text{ and}$$

$$\mathcal{C}_{n+1} := \{\text{finite unions and complements of sets from } \mathcal{C}_n\}.$$

(b) $$\langle \mathcal{C} \rangle_\sigma = \bigcup_{\alpha \in \omega_1} \mathcal{C}_\alpha, \quad \text{where } \mathcal{C}_0 := \mathcal{C} \text{ and for } \alpha > 0,$$

$\omega_1 :=$ the first uncbl cardinal

$$\mathcal{C}_\alpha := \{\text{ctbl unions and complements of sets from } \bigcup_{\beta < \alpha} \mathcal{C}_\beta\}.$$

Proof. (a) By induction on n, it follows that each $\mathcal{C}_n \subseteq \langle \mathcal{C} \rangle$. Thus, it remains to verify that $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ is an algebra, which follows from the fact that if $A, B \in \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, then $\exists n$ such that $A, B \in \mathcal{C}_n$, so $A \cup B, A^c, B^c \in \mathcal{C}_{n+1}$. Details left as HW.

(b) The proof is similar, using transfinite induction and the fact that the supremum of ctbly many ctbl ordinals is a ctbl ordinal. Details left as optional HW. $\square$

Def. Let X be a metric space (more generally, a topological space). The **Borel σ-algebra** is the σ-algebra generated by the open sets. It is denoted by $\mathcal{B}(X)$ and the sets in it are called **Borel sets**.

Def. A *measurable space* is a set equipped with a σ-algebra of its subsets, i.e. a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra.

Measures.

Def. For an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, a map $\mu : \mathcal{A} \to [0, \infty]$ is said to be

- *finitely additive* if $\mu\left(\bigsqcup_{n<k} A_n\right) = \sum_{n<k} \mu(A_n)$

  for any pairwise disjoint $A_1, A_2, \dots, A_k \in \mathcal{A}$.

- *countably additive* if $\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$

  for any pairwise disjoint $A_1, A_2, \dots \in \mathcal{A}$ with $\bigsqcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

(Note that if $\mathcal{A}$ is a σ-algebra then $\bigsqcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ is automatic.)

Def. A *measure* on a measurable space $(X, \mathcal{S})$ is a ctbly additive function $\mu : \mathcal{S} \to [0, \infty]$ mapping $\emptyset$ to 0.

Caution. There is a term *finitely additive measure*, which doesn't mean "measure with extra properties", but rather it is a finitely additive function on an algebra mapping $\emptyset$ to 0.

A measure μ on $(X, \mathcal{S})$ is called

- *finite* if $\mu(X) < \infty$.
- *probability* if $\mu(X) = 1$.
- *σ-finite* if $\exists$ partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ such that $\mu(X_n) < \infty$ $\forall n$.

In practice, we mainly consider σ-finite measures.

Examples. (a) For a set X, the Dirac measure (or point measure) at $x_0 \in X$ is the measure $\delta_{x_0} : \mathcal{P}(X) \to \{0,1\}$ defined by

$$\delta_{x_0}(Y) := \begin{cases} 1 & \text{if } Y \ni x_0 \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } Y \in \mathcal{P}(X).$$

(b) For a set X, the counting measure on X is the measure $\mu : \mathcal{P}(X) \to [0,\infty]$ defined by

$$\mu(Y) := \begin{cases} |Y| & \text{if } Y \text{ is finite} \\ \infty & \text{otherwise} \end{cases}, \quad \text{for } Y \in \mathcal{P}(X).$$

(c) For an unctbl set X, let $\mathcal{S}$ be the σ-algebra of ctbl and co-ctbl sets. Define $\mu : \mathcal{S} \to \{0,1\}$ by

$$\mu(Y) := \begin{cases} 1 & \text{if } Y \text{ is co-ctbl} \\ 0 & \text{if } Y \text{ is ctbl} \end{cases}, \quad \text{for } Y \in \mathcal{S}.$$

This is a probability measure.

Obs. (a) A weighted ctbl sum of measures is a measure. More precisely, if $\mu_0, \mu_1, \mu_2 \ldots$ are measures on a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(X)$ and $w_0, w_1, w_2, \ldots$ are non-negative scalars then $\sum_{n \in \mathbb{N}} w_n \mu_n$ is a measure on $\mathcal{S}$.

(b) A convex combination of probability measures is a probability measure. More precisely, if $\mu_0, \mu_1, \ldots$ are prob. measures on a σ-alg $\mathcal{S} \subseteq \mathcal{P}(X)$ and $w_0, w_1, \ldots$ are non-negative scalars with $\sum_{n \in \mathbb{N}} w_n = 1$, then $\sum_{n \in \mathbb{N}} w_n \mu_n$ is a prob. measure.

In particular:

Examples (continued). (d) A weighted sum of Dirac measures is a measure, i.e. for $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{w_n\}_{n\in\mathbb{N}} \subseteq [0,\infty)$,

$$\sum_{n\in\mathbb{N}} w_n \delta_{x_n} \text{ is a measure.}$$

Note that if X is ctbl, then the counting measure on X is just $\sum_{x\in X} \delta_x$.

Def. Let $\mathcal{S}$ be a σ-alg on a set X containing all singletons. We say that a measure μ on $(X, \mathcal{S})$ is

- atomic if it has atoms.
- purely atomic if μ = ctbl weighted sum of Dirac measures.
- atomless or nonatomic (or continuous, but this is outdated terminology) if it has no atoms.

In the examples above, (a), (b), and (d) are purely atomic, while (c) is atomless. In order to define more useful nonatomic measures, say on $2^{\mathbb{N}}$ or $\mathbb{R}^d$, we need to first define them on small algebras and then extend to the generated σ-algebras. We call a measure on an algebra a premeasure, to emphasize the fact that it is not define on a σ-algebra.

Bernoulli premeasures on $2^{\mathbb{N}}$. Let $\mathcal{A}$ denote the algebra of clopen subsets of $2^{\mathbb{N}}$. By HW,

$$\mathcal{A} = \{\text{finite unions of cylinders}\} = \{\text{finite disjoint unions of cylinders}\},$$

where the last = is because any two cylinders are either disjoint or nested.

Fix $p \in (0,1)$. We define the Bernoulli(p) premeasure $\mu_p : \mathcal{A} \to [0,1]$ as follows:

(i) For a word $w \in 2^{<\mathbb{N}}$, put $\tilde{\mu}_p([w]) := p^{n_1} \cdot (1-p)^{n_0}$, where $n_0 :=$ # of 0s in w, $n_1 :=$ # of 1s in w.

(ii) For $A \in \mathcal{A}$, write A as a finite disjoint union $\bigsqcup_{n<k} C_n$ of cylinders, and put
$$\mu_p(A) := \sum_{n<k} \tilde{\mu}_p(C_n).$$

We first need to show that μ_p is well-defined, i.e. doesn't depend on the choice of the partition $A = \bigsqcup_{n<k} C_n$.

Terminology. The base of a cylinder $C \subseteq 2^{\mathbb{N}}$ is the unique word $w \in 2^{<\mathbb{N}}$ such that $C = [w]$.

Claim (a). For any finite word $w \in 2^{<\mathbb{N}}$ and $\ell \geq 0$,
$$\tilde{\mu}_p([w]) = \sum_{u \in 2^\ell} \tilde{\mu}_p([wu]).$$

Proof. It is enough to prove for $\ell := 1$ and apply induction (on ℓ). But for $\ell = 1$, we have
$$\sum_{u \in 2} \tilde{\mu}_p([wu]) = \tilde{\mu}_p([w0]) + \tilde{\mu}_p([w1]) = \tilde{\mu}_p([w]) \cdot (1-p) + \tilde{\mu}_p([w]) \cdot p = \tilde{\mu}_p([w]).$$
$\square$